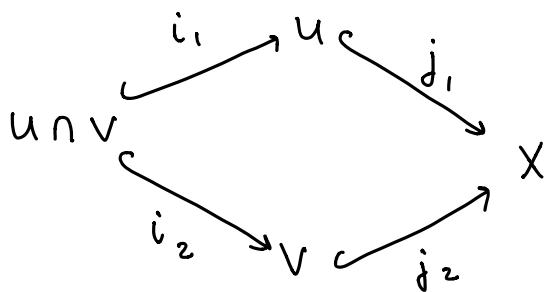


The Seifert-van Kampen Theorem

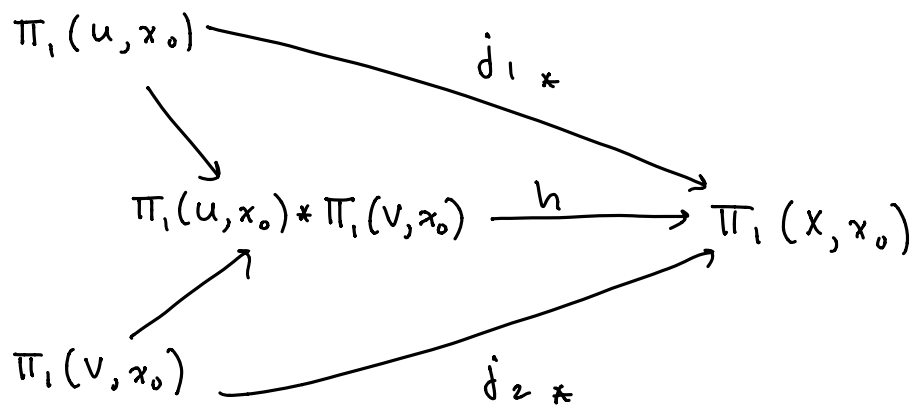
We will only state the classical formulation of the theorem, without proof, and use it to compute some fundamental groups. For details, see section 70 of Munkres.

Let $X = U \cup V$, where U and V are open in X . Assume $U, V, U \cap V$ are path connected. Let $x_0 \in U \cap V$.

Consider the following inclusions:



Let h be the unique homomorphism s.t.



commutes.

Thm: (Seifert-van Kampen) The homomorphism h defined above is surjective, and the kernel N is the smallest

normal subgroup containing words of the form

$$(i_{1*}(g)^{-1}, i_{2*}(g)) \text{ for } g \in \pi_1(U \cap V, x_0).$$

i.e. $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) / N.$

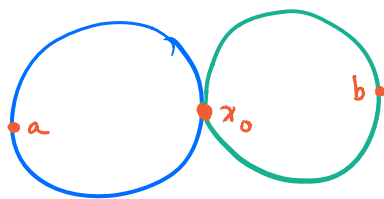
Cor: If $U \cap V$ is simply connected, then $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0).$

Cor: If $\pi_1(V, x_0)$ is trivial, then $\pi_1(X, x_0) \cong \pi_1(U, x_0) / N,$

where N is the smallest normal subgroup containing the image of

$$i_{1*} : \pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0).$$

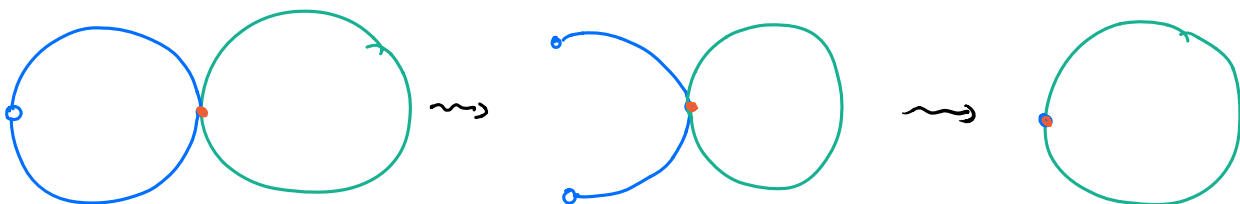
Ex: Consider the figure-eight space X



We can give X the subspace topology in $\mathbb{R}^2.$

Let a and b be the antipodal points to x_0 in each circle, respectively. Let $U = X \setminus \{a\}, V = X \setminus \{b\}.$

Then U and V both have a deformation retraction onto a circle:



(Each half circle is homeomorphic to a convex subspace in \mathbb{R}^2)

and $U \cap V$ is contractible. Thus, N is trivial, so

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) \cong \mathbb{Z} * \mathbb{Z} \quad \text{and it's}$$

generated by the loops generating the fund. gps of each S^1 .

Wedges of circles

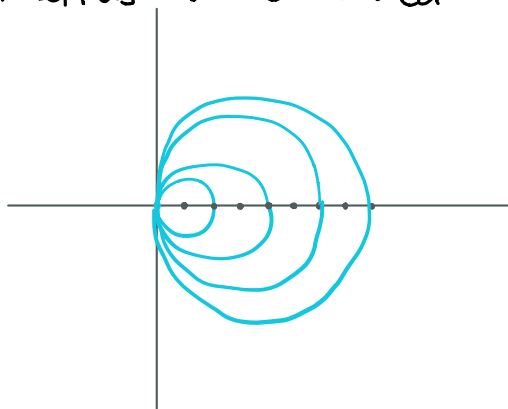
We can adjoin more than two circles together as follows:

Def: Let X be a Hausdorff space that is the union of S_1, \dots, S_n , and S_i is homeomorphic to $S^1 \quad \forall i$. Suppose $\exists p \in X$ s.t. $S_i \cap S_j = \{p\} \quad \forall i \neq j$. X is called the wedge of the circles S_1, \dots, S_n , and is denoted

$$X = S_1 \vee S_2 \vee \dots \vee S_n.$$

Note: Each S_i is compact, and thus closed in X .

Also, X is homeomorphic to $C_1 \cup \dots \cup C_n \subseteq \mathbb{R}^n$, where C_i is the circle of radius i centered at $(i, 0)$:



We can calculate the fundamental group of the wedge of n

circles by induction:

Thm: If $X = S_1 \vee \dots \vee S_n$, the wedge of n circles, and f_i is a loop generating $\pi_1(S_i, x_0)$, then

$\pi_1(X, x_0)$ is the free group on generators f_1, \dots, f_n .

Pf: Thm holds for $n=1$. Assume true for $n-1$.

For each i , choose $q_i \in S_i$ s.t. $q_i \neq x_0$.

Let $U = X \setminus \{q_1, \dots, q_{n-1}\}$ and $V = X \setminus \{q_n\}$.

Then U has a deformation retraction onto S_n , and

V onto $S_1 \vee \dots \vee S_{n-1}$. $U \cap V$ is contractible.

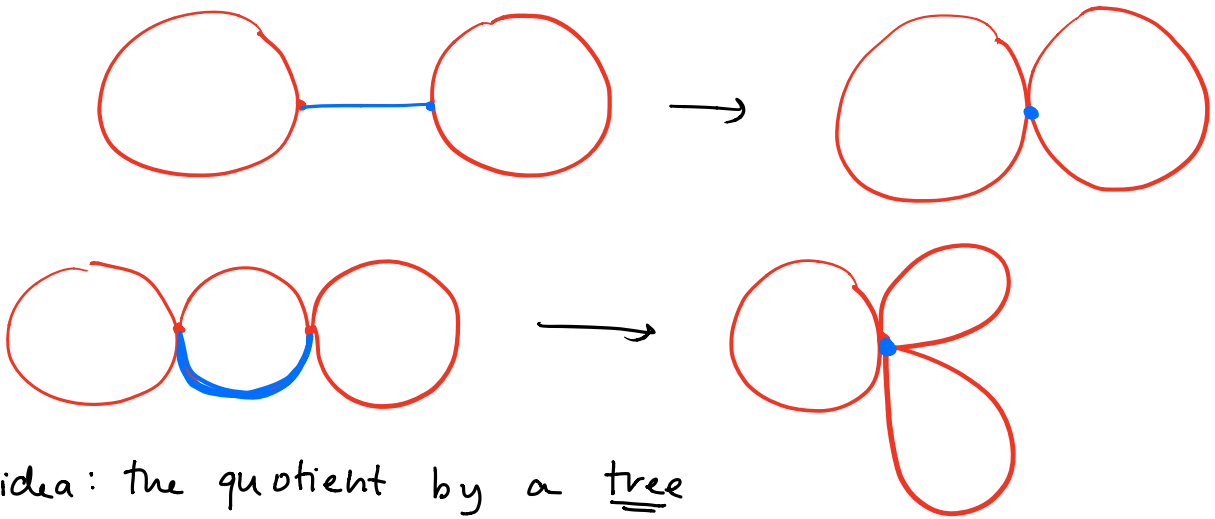
Thus, $\pi_1(X, x_0) \cong \pi_1(S_1 \vee \dots \vee S_{n-1}, x_0) * \pi_1(S_n, x_0)$, which is

the free product of the free group on $n-1$ generators w/
the free group on $n-1$ generators. \square

It turns out any graph has the homotopy type of the wedge of circles

Claim: If X is a finite graph (homeomorphic to the disjoint union of I 's glued at the endpoints) then X has the homotopy type of the wedge of finitely many circles.

Ex:



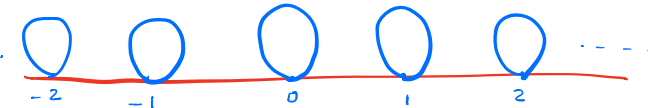
Rough idea: the quotient by a tree
(graph w/ no loops) is a homotopy equivalence.

We can also define an infinite wedge of circles, indexed by $\mathbb{Z}_{>0}$:

$X = \bigcup_{n \in \mathbb{Z}_{>0}} S_n$ where S_n is the circle of radius n centered at $(n, 0)$.

Exercise: 1.) If f_n is the loop generating $\pi_1(S_n, (0,0))$, then $\pi_1(X, (0,0))$ is the free group on generators $\{f_1, f_2, \dots\}$.

2.) X is not homeomorphic to $\bigcup_{n \in \mathbb{Z}_{>0}} C_n$, where C_n is the circle of radius $1/n$ centered at $(1/n, 0)$.

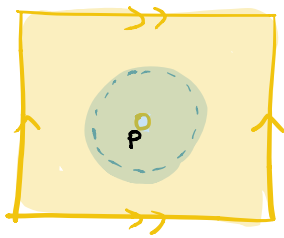
3.) The familiar space 

has the same fundamental gp as X .

Fundamental groups of some surfaces

We can calculate the fundamental group of the torus in a

different way now:



$$U = (I \times I) \setminus P$$

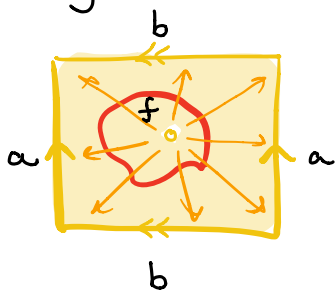
$V =$ open ball of radius $< \frac{1}{2}$ around P

U has homotopy type of wedge of two circles

V is simply connected.

$U \cap V$ has homotopy type of S^1 .

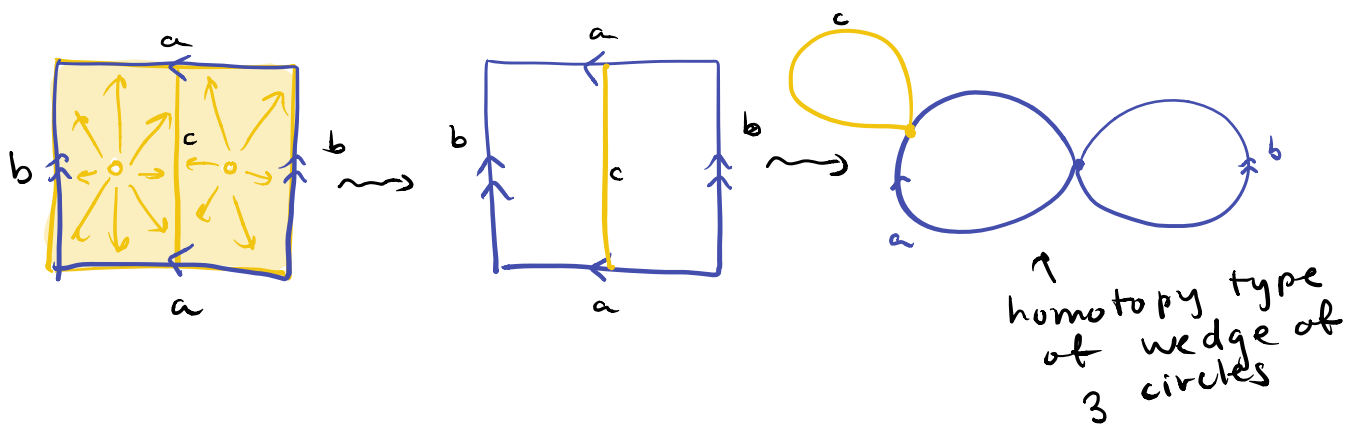
So $\pi_1(S^1 \times S^1) = \pi_1(U) / N$, where N is generated by the image of the loop f



If $\pi_1(U)$ is the free gp on a, b , the image of $[f]$ is $aba^{-1}b^{-1}$, so

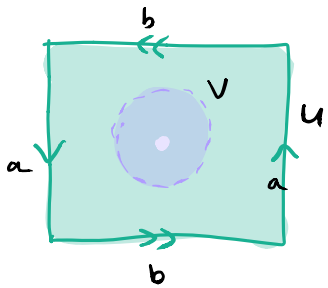
$$\pi_1(S^1 \times S^1) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

What about a torus minus two points?

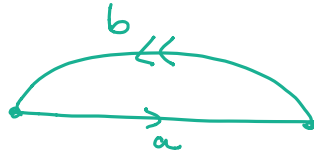


↑ homotopy type of wedge of 3 circles

We can also calculate $\pi_1(\mathbb{R}P^2)$ like this:



We can deformation retract U to



whose fund. gp
is the free gp
on one generator, $c=ab$.

and V is simply connected, but now, the image
of a generator of $\pi_1(U \cap V)$ is $abab = c^2$

$$\text{so } \pi_1(\mathbb{R}P^2) \cong \langle c \mid c^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$